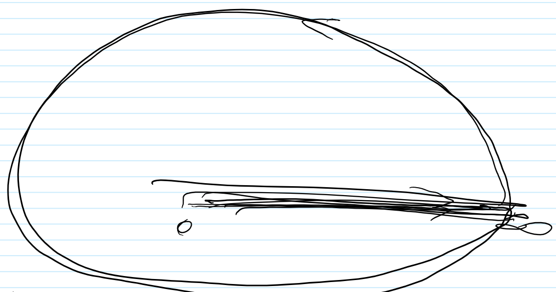
More applications of Residual Calculus

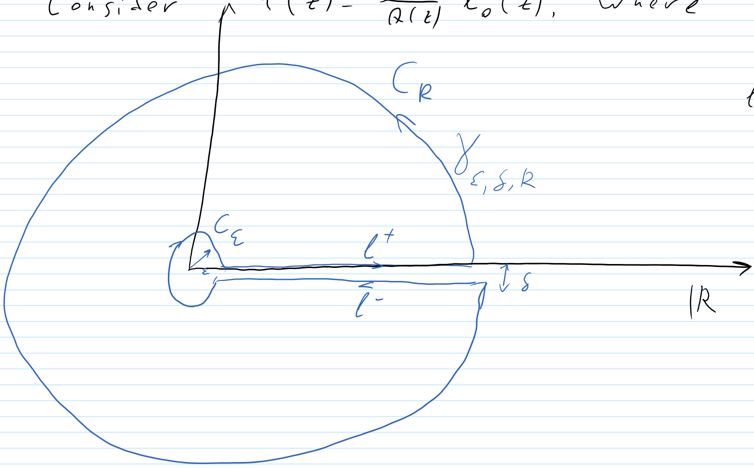
Friday, November 24, 2023 6:06 PM



5) $\int_0^\infty \frac{P(x)}{Q(x)}\,dx$, $\deg Q \ge \deg P + 2$, $\forall x \ge 0,\ Q(x) \ne 0$.

Consider $f(z) = \frac{P(z)}{Q(z)}\, l_0(z)$, where $l_0(z) := \log|z| + \arg_0 z$, $0 \le \arg_0 z < 2\pi$.

$l_0(z)$ – a branch of log discontinuous on $\mathbb{R}_+$.



Take ε, δ – small, R large, so that all $z: Q(z)=0$ lie inside the contour. Then

$$\oint_{\gamma_{\varepsilon,\delta,R}} f(z)\,dz = 2\pi i \left(\sum_{Q(z_j)=0} \operatorname{Res}_{z=z_j} \frac{P(z)}{Q(z)}\, l_0(z) \right)$$

But $\oint_{l^+} f(z)\,dz = \int_\varepsilon^R \frac{P(x)}{Q(x)}\, l_0(x)\,dx$

$\oint_{l^-} f(z)\,dz = -\int_\varepsilon^R \frac{P(x-i\delta)}{Q(x-i\delta)}\, l_0(x-i\delta)\,dx$

$\boxed{\text{Fix } \varepsilon, R.}$ Let $\delta \to 0$. Then $\frac{P(x-i\delta)}{Q(x-i\delta)} \to \frac{P(x)}{Q(x)}$, uniformly on $[\varepsilon, R]$.

$l_0(x-i\delta) \to \underline{l_0(x) + 2\pi i}$, also uniformly on $[\varepsilon, R]$.

So, as $\delta \to 0$, $\oint_{l^+} f(z)\,dz + \oint_{l^-} f(z)\,dz = \int_\varepsilon^R \frac{P(x)}{Q(x)}\, l_0(x)\,dx -$

$2\pi i \int_\varepsilon^R \frac{P(x)}{Q(x)}\,dx - \int_\varepsilon^R \frac{P(x)}{Q(x)}\, l_0(x)\,dx =$

$-2\pi i \int_\varepsilon^R \frac{P(x)}{Q(x)}\,dx$.

So $\oint_{C_R} f(z)\,dz - \oint_{C_\varepsilon} f(z)\,dz - 2\pi i \int_\varepsilon^R \frac{P(x)}{Q(x)}\,dx = 2\pi i \left(\sum_{Q(z_j)=0} \operatorname{Res}_{z=z_j} \frac{P(z)}{Q(z)} \ell_0(z) \right)$

As $R \to \infty$

$$\left| \oint_{C_R} f(z)\,dz \right| \le 2\pi R \cdot \log R \max_{|z|=R} \left| \frac{P(z)}{Q(z)} \right| \to 0.$$

$$\underbrace{\quad}_{\le \frac{C}{R^2}}$$

As $\varepsilon \to 0$

$$\left| \int_{C_\varepsilon} f(z)\,dz \right| \le 2\pi\varepsilon \log \tfrac{1}{\varepsilon} \max_{|z|\le\varepsilon} \left| \frac{P(z)}{Q(z)} \right| \to 0.$$

So we get

$$\boxed{\int_0^\infty \frac{P(x)}{Q(x)}\,dx = -\left(\sum_{Q(z_j)=0} \operatorname{Res}_{z=z_j} \frac{P(z)}{Q(z)} \ell_0(z) \right)}$$

6) $\int_0^\infty x^\alpha \frac{P(x)}{Q(x)}\,dx \qquad 0<\alpha<1, \qquad \deg Q \ge \deg P + 2$ (to converge).
$Q(x) \ne 0 \quad x>0.$

As in 5): define $\ell_0(z)$, $z^\alpha := e^{\alpha \ell_0(z)}$.
Use the same contour for

$$f(z) = z^\alpha \frac{P(z)}{Q(z)}$$

$$\oint_{\gamma_{\varepsilon,\delta,R}} f(z)\,dz = 2\pi i \left(\sum_{Q(z_j)=0} \operatorname{Res}_{z=z_j} \frac{P(z)}{Q(z)} z^\alpha \right).$$

$$\oint_{\ell^+} f(z)\,dz = \int_\varepsilon^R x^\alpha \frac{P(x)}{Q(x)}\,dx$$

but $\lim_{\delta\to 0} \oint_{\ell^-} \underbrace{(x-i\delta)^\alpha}_{\downarrow} \frac{P(x-i\delta)}{Q(x-i\delta)}\,dx = -\int_\varepsilon^R e^{2\pi i\alpha} x^\alpha \frac{P(x)}{Q(x)}\,dx.$

$x^{\alpha} e^{2\pi i \alpha}$

So, since, as before, $\lim\limits_{\varepsilon \to 0} \oint\limits_{C_\varepsilon} f(z)\,dz = 0$,

$\lim\limits_{R \to \infty} \oint\limits_{C_R} f(z)\,dz = 0$

$$\int_0^\infty x^\alpha \frac{P(x)}{Q(x)}\,dx = \frac{2\pi i}{1 - e^{2\pi i \alpha}} \Big(\sum_{Q(z_j)=0} \operatorname{Res}_{z=z_j} \frac{P(z)}{Q(z)} z^\alpha \Big).$$

$$\frac{\pi}{\sin \pi\alpha \, e^{\pi i \alpha}}$$

7) Let us consider

$\sum\limits_{n=0}^{\infty} \frac{P(n)}{Q(n)}$, where P, Q - polynomials, $\deg Q \geq \deg P + 2$, $Q(n) \neq 0$ ($n \in \mathbb{Z}$).

(Example: $\sum\limits_{n=0}^{\infty} \frac{1}{n^2+1}$)

Assume $\frac{P(z)}{Q(z)}$ - even $\left(\frac{P(-z)}{Q(-z)} = \frac{P(z)}{Q(z)} \right)$.

Or we can consider $\sum\limits_{n=-\infty}^{\infty} \frac{P(n)}{Q(n)}$.

Consider $f(z) = \frac{P(z)}{Q(z)} \pi \cot \pi z$.

<u>Observerve:</u> $\pi \cot \pi z = \pi i \left(\frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}} \right) = \pi i \left(\frac{e^{2\pi i z} + 1}{e^{2\pi i z} - 1} \right)$

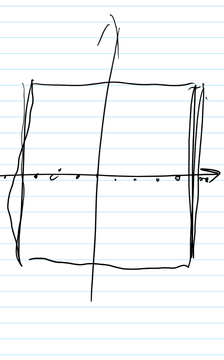
Singularities: at $z = n$, $n \in \mathbb{Z}$.

$$\operatorname{Res}_{z=n} \frac{P(z)}{Q(z)} \pi \cot \pi z = \frac{\pi f(n) \cos \pi n}{(\sin \pi z)'|_{z=\pi n}} = \frac{P(n)}{Q(n)}.$$

Other singularities of $f(z)$ - zeroes of Q.

Consider a square $S_N = \{ -N - \frac{1}{2} \leq \operatorname{Re} z \leq N + \frac{1}{2},\ -N - \frac{1}{2} \leq \operatorname{Im} z \leq N + \frac{1}{2} \}$

Observe that $w \to \frac{w+1}{w-1}$ maps 1 to ∞. It is one-to-one,



so the region $|w-1| > \delta$ is mapped to a bounded region (some disk). On ∂S_N: $|e^{2\pi i z} - 1| > 1 - e^{-\pi}$ (consider each side).

So $\exists\, M$: $|\pi \cot \pi z| \leq M$ $\forall z \in \partial S_N$.

$$\text{So } \Big| \oint_{\partial S_N} f(z)\,dz \Big| \leq \ell(\partial S_N) \cdot M \cdot \max_{\partial S_N} \frac{P(z)}{Q(z)} \leq 4(N+1) M \cdot \frac{C}{N^2} \to 0.$$

So $\left|\oint_{\partial S_N} f(z)\,dz\right| \le \ell(\partial S_N)\cdot M \cdot \max_{\partial S_N} \frac{P(z)}{Q(z)} \le 4(N+1)M\cdot\frac{C}{N^2} \to 0.$

for some C.

So $\lim_{N\to\infty} \frac{1}{2\pi i}\oint_{\partial S_N} f(z)\,dz = \overbrace{\sum_{z\in\mathbb{Z}} \operatorname{Res}_{z=n} f(z)}^{P(n)/Q(n)} + \sum_{Q(z_j)=0} \operatorname{Res}_{z=z_j} f(z).$

So $\displaystyle\sum_{n=-\infty}^{\infty} \frac{P(n)}{Q(n)} = -\sum_{Q(z_j)=0} \operatorname{Res}_{z=z_j} f(z)$

For even $\frac{P(z)}{Q(z)}$:

$$\sum_{n=0}^{\infty}\frac{P(n)}{Q(n)} = \frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{P(n)}{Q(n)} + \frac{1}{2}\frac{P(0)}{Q(0)} = \frac{1}{2}\frac{P(0)}{Q(0)} - \sum_{Q(z_j)=0}\operatorname{Res}_{z=z_j} f(z)$$

For our example, $z_1 = i$, $z_2 = -i$

$$\operatorname{Res}_{z=i} \frac{\pi\cot\pi z}{z^2+1} = \frac{\pi}{2}\left(\frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}}\right)$$

$$\operatorname{Res}_{z=-i} \frac{\pi\cot\pi z}{z^2+1} = \frac{\pi}{2}\left(\frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}}\right)$$

So $\boxed{\displaystyle\sum_{n=0}^{\infty}\frac{1}{n^2+1} = \frac{\pi}{2}\left(\frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}}\right) + \frac{1}{2}.}$